THE UNIVERSALITY OF NONLINEAR SYSTEMS AND THEIR DESCRIPTION IN THE AREA OF MODERN PRODUCTION

TOMAS STEJSKAL[1], JOZEF SVETLIK[1], JOZEF DOBRANSKY[2]

[1]Technical University of Kosice, Faculty of Mechanical Engineering, Department of Production Systems and Robotics, Kosice, Slovak Republic

[2]Technical University of Kosice, Faculty of Manufacturing Technologies, Department of Production Management, Kosice, Slovak Republic

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jozef.svetlik@tuke.sk

The article discusses the five-element modelling of a nonlinear system. In most cases, the nonlinearity is resolved between two objects that interact with each other. For example, the relationship between production and maintenance is non-linear. If the production system is inherently reliable, the maintenance requirements are lower. Such a system is often inefficient and expensive. On the other hand, a more powerful but less reliable system requires higher maintenance requirements. The system is cheaper, but costs are higher for maintenance. These are dynamically changing systems, where it is necessary to find an appropriate balance between maintenance costs and production costs at every moment. In the background of such non-linear systems are mathematical models which, when simplifying relationships, have five objects with the same properties. Mathematical objects are supported in some relationships and suppressed in others. Together, however, they create a stable dynamic system that can be used to model, among other things, the production relations of companies.

KEYWORDS

Non-linear system, five elements, production, maintenance, modelling

1 INTRODUCTION

Non-linear systems have a significantly more frequent occurrence in various fields of technology as well as social relations. Basically, purely linear systems are just a mathematical abstraction. We can describe and predict the behaviour of linear systems quite simply and precisely. For slightly nonlinear systems, there is the effort to make the solution linear. In simple cases, the calculation gives an acceptable accuracy of the estimation of the behaviour of the systems. In strongly nonlinear systems, the interrelationships of objects are unclear and mathematical prediction is theoretically impossible [Trojanova 2021]. A system that has entered the chaotic region can only be predicted for a few steps. After that, the accuracy of the prediction is lost. A typical example is a meteorological weather forecast. Today's technology can successfully predict developments within ten days. But what will happen in a hundred days is impossible to calculate analytically. In this regard, the only source of information is the simulation, which gives the general properties of the system, such as the gatekeepers of the definition field, the prevailing form of behaviour, and the like. In 1889, the famous French mathematician Henry Poincare proved the intractability of the three-body problem [Barrow-Green 1997, Musielak 2014]. The essence is that the result of a solution does not depend on the accuracy of the initial conditions. In other words, by increasing the accuracy, the result does not converge to some definite course of development of the monitored function.

Nonlinear systems again became the centre of interest in the early 60s, when rapidly developing computer technology became available. The theory of chaos appeared and the related theory of fractals.

Nonlinear systems have been extensively studied in the field of market dynamics [Man 2011, Modrak 2017]. Mainly in the supply and demand relationship, where small changes in prices can lead to large changes in the quantity of goods sold. Population models for the description of biological processes have appeared. Non-linear relationships within ecosystems have begun to be explored in detail.

The need for a better description of nonlinearities also arose in physical systems [Saga 2019]. It happened similarly in the fields of biology and chemistry. Social systems are full of non-linear relationships. For example, interactions between individuals in a crowd, where small changes in the behaviour of one individual can affect the behaviour of the entire crowd.

2 INTERNAL AND EXTERNAL INFLUENCES CAUSING STRONG NONLINEARITY

The simplest strongly nonlinear system is a simple logistic equation generating chaotic behaviour of populations. This process is extensively researched. It is a relationship between two entities that influence each other. Populations grow or decline depending on the current situation in the environment. Depending on the setting of the control parameter r, populations can behave differently. The parameter r is the reproduction rate of the monitored population. The logistic equation has a very simple structure and is suitable for simulating various regimes. The behaviour of a simple ecological system, for example the relationship between the size of grassland and the number of a hare population, can be written using a recurrent logistic equation

$$x_{n+1} = r x_n (1 - x_n) , \qquad (1)$$

where x_n is the current population size and r is a control parameter that corresponds to the rate of reproduction. The control parameter r changes the dynamics of the development of the nonlinear system. If the rate of reproduction of the population is slow, then the population disappears. If it is moderately fast, the population will stabilize at a certain size. If the parameter r increases, the population starts to behave cyclically. Steady population sizes alternate regularly. At the bifurcation point, the population size begins to double first.

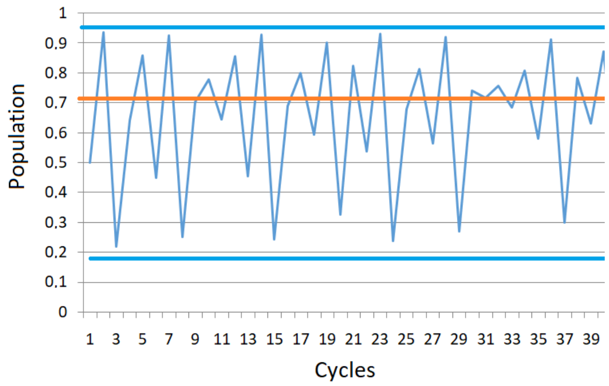

Figure 1. Chaotic course of the logistic equation for r= 3.75

By further increasing the parameter r, the number of bifurcation points increases and the cycles multiply, until finally the population begins to behave chaotically. With an extreme increase in the parameter r, divergence occurs, and the population disappears after a single cycle. With the parameter $r = 3.75$, the development of the number of the population clearly has a chaotic character. The population size randomly oscillates around the neutral axis. The span is closed between two guardrails (Figure 1).

The concept of population is used from a historical context. In fact, any object that can change is a form of population. A chaotic system can only arise when strongly non-linear relationships between objects are applied. This is a special case of nonlinearity, which is convenient to describe verbally. A chaotic system of population behaviour in real nature is very rare. It appears, for example, in the behaviour of hamsters [Juskaitis 1999].

Common populations occur in a stable area. Nevertheless, certain cyclical behaviour can be observed even in stable populations. The change in the size of the population is given, unlike the chaotic equation, by external influences. In this direction, a significant change in the abundance of the lynx (Lynx canadensis) and different prey species such as the snowshoe hare (Lepus americanus) has been observed. This one of the longest observations was compiled by Charles Elton in 1937 and was documented in records of the number of furs brought to the Hudson Bay Company [Zhang 2007] (Figure 2).

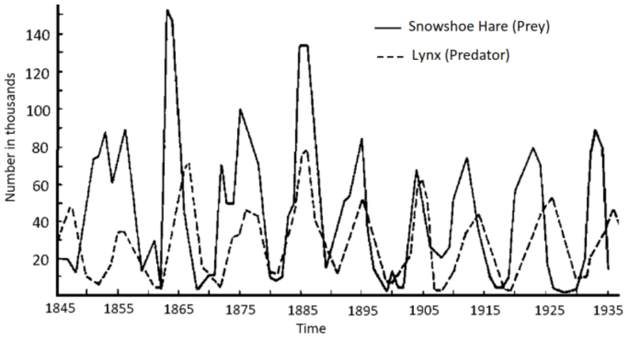

Figure 2. Fluctuations in the number of lynx and hare populations

Based on the graph, it could be assumed that hare and lynx populations interact. This is not the case. In general, predators have many times lower chances of survival than hunted animals. Fluctuations in populations are caused by the amount of food available for herbivores. Similarly, the number of predators is dependent on the amount of available food in the form of herbivores. So, it is a one-way dependency. The cyclical behaviour of the population, regardless of the real causes, is given by non-linear relationships. It does not matter whether the cause of the changing cycles is given by internal or external conditions. The work of Stejskal [Stejskal 2011] dealt with the initial, less analytical procedure of describing the five elements. Here is a more accurate and well-crafted analysis.

3 A NEW METHOD OF SYSTEM NONLINEARITY ANALYSIS

A new approach to the analysis of nonlinear systems is a formal description of the systems regardless of causality. From a certain point of view, it could be debated whether it is necessary to know the causal connection of the change of the object. Every change in an object has its internal and external causes. If we look at the system globally, it is always a matter of a certain balancing of forces between interacting elements. Thus, the movement of growth or decline of an object is ultimately determined by the extent to which the system has balanced power relations with its surroundings. Paradoxically, we do not need to know the causal connections, despite the fact that there are always causal connections that cause the given change.

Let's take, for example, the aforementioned logistic equation and try to verbally describe the behaviour of this system in the chaotic mode of progress. Verbal description ensures tracking of external signs of non-linearity. This can be shown to be of importance in the construction of multi-element systems that have specific relationships [Dyadyura 2021].

3.1 *Verbal description of a chaotic system*

First of all, it can be stated that the cycles change only in certain parameters.

- A small object tends to quickly grow to an above-average value. At the same time, it has a tendency of slow decline.
- When the average value is reached, the object loses the tendency of rapid growth. It may grow slowly but acquires a tendency to decline rapidly.
- A large object tends to grow slowly and shrink quickly. From this state, it can easily acquire an average value, but also an extremely small value.
- With a rapid decrease, the object reaches an extreme value again and becomes small.
- Close to the average size, the tendencies to change the size are the smallest. On the other hand, but with great probability, there will be a fluctuation that will generate either an extremely large or an extremely small object.

This is how the size of the object changes alternately. Speech is chaotic. The verbal description points to a significant instability of the system.

The new analysis is based on the assumption of the existence of simple control functions in relation to the monitored object. In the previous verbal description, we encountered the function of growth and decrease. We also encountered a fast and slow tendency of change. Based on simple Boolean algebra, four basic relationships of an object with its surroundings are practically possible (Figure 3).

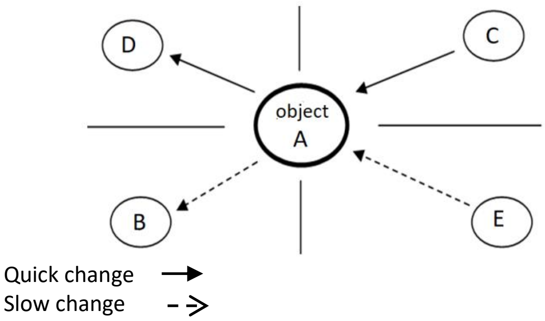

Figure 3. Basic non-linear relationships of the object with its surroundings

3.2 *Model 1: Development in the direction of slow decline A, E, D, C, B*

From Figure 3, the following relationships affecting object A can be determined:

- Fast growth C
- Slow growth E
- Rapid decline D
- Slow decline B

The surroundings of the object with which the object creates relationships can be viewed as an object-object relationship. A minimum of 5 objects are required to provide four simple relationships. For the construction of a non-linear system, it is advisable to specify the requirement of equivalence of objects. The smallest possible number of object connections naturally creates a pentagram of relationships. It should be emphasized that the decomposition into elementary relations may not correspond to a real object, but it can still be done. Any strongly nonlinear system can be decomposed into five objects with four basic relations in the above manner.

The following non-linear equation provides the required relationships to object A based on Figure 3.

$$A_{n+1} = A_n + \frac{C_n}{A_n^2} r - \frac{B_n}{A_n}, \tag{2}$$

where r is a control parameter, C_n is an object that ensures the rapid growth of object A (due to the positive sign and the quadratic term in the denominator of the fraction), B_n is the object that ensures the slow decline of object A (due to the negative sign and the linear term in the denominator of the fraction). In total, there are five non-linear equations, which also complement each other with inverse relations. That is, slow growth and rapid decline. The following equations alternate the positions of their objects cyclically.

$$B_{n+1} = B_n + \frac{D_n}{B_n^2} r - \frac{C_n}{B_n}, \tag{3}$$

$$C_{n+1} = C_n + \frac{E_n}{C_n^2} r - \frac{D_n}{C_n}, \tag{4}$$

$$D_{n+1} = D_n + \frac{A_n}{D_n^2} r - \frac{E_n}{D_n}, \tag{5}$$

$$E_{n+1} = E_n + \frac{B_n}{E_n^2} r - \frac{A_n}{E_n}. \tag{6}$$

So, we got five equivalent non-linear recurrent equations. Individual relationships can also be displayed graphically (Figure 4).

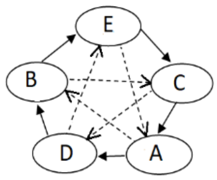

Figure 4. Pentagram of basic non-linear relationships

In order to start up relationships and sizes, it is necessary to specify input values at the beginning, which will ensure the start-up of the system to the established proportions of object sizes. Regardless of the specific values of the input data, the individual parameters of the equations will always stabilize cyclically and maintain stable mutual ratios. Although these ratios change due to internal development, they are in a fixed functional relationship. The individual ratios alternate cyclically with increasing steps, and it is a lawful process that takes place without external influence. With a significantly large control parameter, changes in ratios take place slowly. In that case, we can claim that the model simulates a steady state.

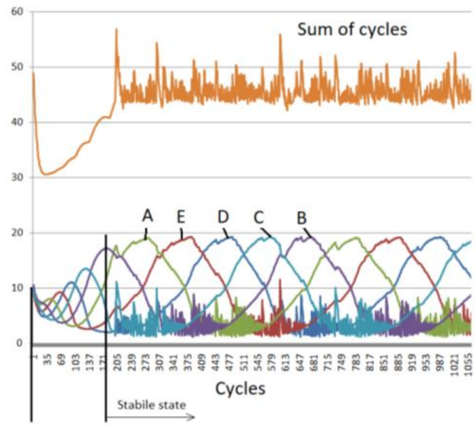

Figure 5. Alternation of cycles and size of objects in model 1

It can be verified by simulation that the control parameter must have a value greater than 3.5 $(r = 3.5)$. At a smaller value, the system loses stability and breaks down. Individual cycles alternate regularly in the direction of slow decline (A, E, D, C, B). This can be identified on the graph (Figure 5).

The interesting thing about this analysis is that the arrangement of mutual relations between individual objects is legal and does not depend on the value of the input conditions at the start of the cycles. It is about achieving the highest efficiency of cycles, assuming equivalent objects.

3.3 Separate development cycle of object in model 1

The selected individual object goes through different development phases (Figure 6). The first phase is the progressive growth of the object (1). Depending on the circumstances, this growth can be intense or slow. The second stage is a stable prosperous state (2). The given object is dominant in the given environment. The third phase is regressive decline (3). Depending on the circumstances, the loss can be intense or slow. The fourth stage is an unstable state with many unexpected fluctuations (4). The object thrives thanks to considerable variability and adaptation to the environment. After this phase, the cycle repeats again.

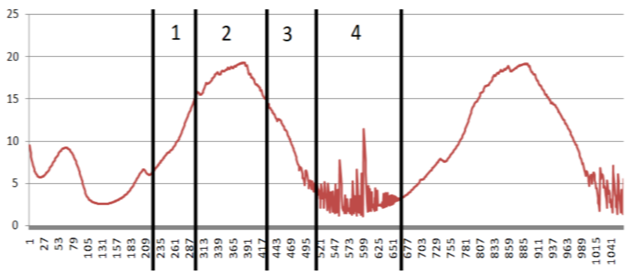

Figure 6. Separate development cycle of one object in model 1

3.4 Model 2: Development in the direction of rapid growth A, D, B, E, C

Changing the system is also possible in a different order of elements. In the second model, the rapid change during the growth of the object is dominant. The following non-linear equation provides the required relationships to object A based on Figure 4.

$$A_{n+1} = A_n + \frac{B_n}{A_n^2} r - \frac{C_n}{A_n}, \tag{7}$$

where r is a control parameter, B_n is an object that ensures the rapid growth of object A (due to the positive sign and the quadratic term in the denominator of the fraction), C_n is the object that ensures the slow decline of object A (due to the negative sign and the linear term in the denominator of the fraction). The following equations alternate the positions of their objects cyclically.

$$B_{n+1} = B_n + \frac{C_n}{B_n^2} r - \frac{D_n}{B_n}, \quad (8)$$

$$C_{n+1} = C_n + \frac{D_n}{C_n^2} r - \frac{E_n}{C_n}, \quad (9)$$

$$D_{n+1} = D_n + \frac{E_n}{D_n^2} r - \frac{A_n}{D_n}, \quad (10)$$

$$E_{n+1} = E_n + \frac{A_n}{E_n^2} r - \frac{B_n}{E_n}. \quad (11)$$

Individual cycles alternate regularly in the direction of rapid growth (A, D, B, E, C). This can be identified on the graph (Figure 7).

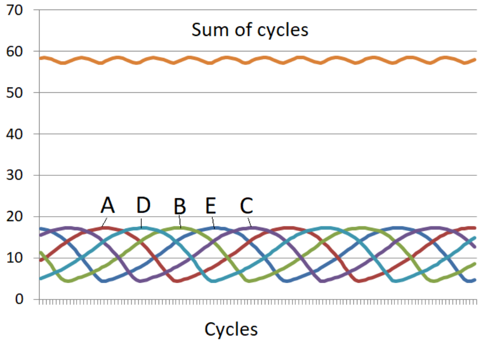

Figure 7. Alternation of cycles and size of objects in model 2

3.5 *Separate development cycle of object in model 2*

The selected individual object goes through different development phases (Figure 8). The first phase is the progressive growth of the object (1). The second stage is a stable prosperous state (2). The given object is dominant in the given environment. The third phase is regressive decline (3). The fourth phase is a stable short-term state (4). After this phase, the cycle repeats again.

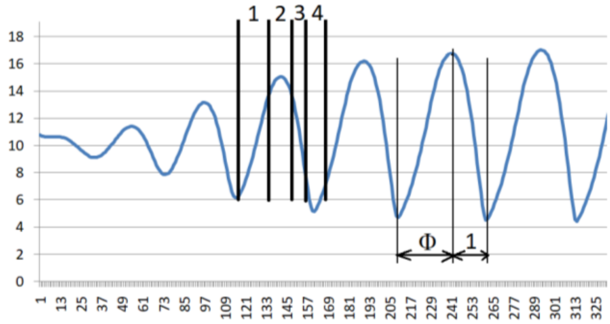

Figure 8. Separate development cycle of one object in model 2

In contrast to the first model, the process of changing the size of objects is smooth and takes fewer steps. In the form of separate cycles, it is even possible to identify the ratio of the golden ration $(\Phi{:}1 \approx 1.618)$, which is a consequence of the pentagram arrangement of the system.

3.6 *Combinations of models 1 and 2*

The combination of models leads to the breakdown of the system. Stability is maintained only in the only case, if we replace a single equation from model 2 in model 1. In that case, the system stops behaving cyclically and maintains the initial ratio of objects (Figure 9).

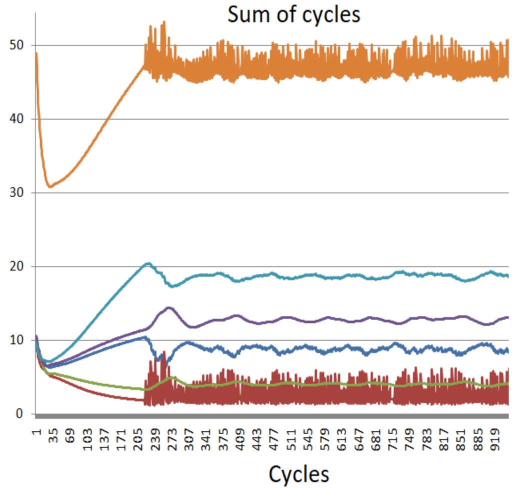

Figure 9. Steady cycles when replacing only the equation in model 1 with the equation from model 2

4 EXAMPLE OF THE SOLUTION OF A NONLINEAR SYSTEM

The five-element system can be applied to any nonlinear system. First, we choose a monitored circle. Subsequently, we will look for influences that support the given object for fast growth, slow growth, and also influences that degrade the given object by rapid decline or slow decline. Maintenance-production relations are well known in technical practice. We can construct a five-element model.

4.1 *The balance of maintenance and production relations*

Let's start from the verbal description of relationships, while we will look for connections with the five-element system. A common situation in the business economy is dealing with machine maintenance costs. It is evident that if the production on the machine park is reliable, then a smaller number of maintenance interventions is necessary [Peterka 2020, Trefilov 2020]. We could call this a steady state. If a company wants to increase productivity, it has several options. For example, the machine park can be expanded, and new employees can be hired. A more likely approach will be to increase the performance of the existing machine park. Such a solution is always connected with higher requirements for operator qualification. At the same time, dynamic forces naturally increase, which reduce the durability of tools and machines over time [Turygin 2018, Bozek 2021]. Thus, the requirements for maintenance interventions are increasing. It is clear from the above remarks that this is a highly dynamic nonlinear system [Kuric 2021].

Let the main monitored object be the size (volume) of maintenance. In general, the size of this object can also be expressed by a cost estimate. The question is what factors reduce or increase the size of this object.

Factor A

- Factor A is the existing level of maintenance.

Factor E causes a slow increase in A

- Training of maintenance personnel. Provision of equipment for technical diagnostics.

Factor D causes a rapid decline in A

- Investing in the modernization of the machine park. Purchase and operation of reliable machines. Reduction of production performance.

Factor C causes a rapid increase in A

- Increasing production performance in the old machine park.

Factor B causes a slow decline in A

- Low qualification of machine operators. Failure to ensure the purchase of spare parts. Personnel reduction of the maintenance status. Inadequate backup systems.

One can think about the relationships shown in terms of development. For example, let the dominant maintenance in the system (factor A) is strongly supported by the possibility of increasing the performance in the old machine park (factor C). If this were not the case, the system will very likely develop in this direction.

The external environment around the object determines the direction of development. That direction is also conditioned by other mutual relationships according to the pentagram. Individual relationships logically follow each other. The analysis has the character of a SWOT analysis. Only instead of strengths and weaknesses, fast and slow growth is used. Slow and fast decay are used instead of opportunities and threats.

5 CONCLUSION

The article points to a new way of modelling nonlinear systems. In this method, the causal connections of the nonlinearity are not followed, but the lawful development of the system that seeks its balance is followed. Such a system is variable and small fluctuations in the environment affect the direction of development and the change of the monitored object.

The non-linearity of the relationships itself is modelled from a linear and a quadratic element. In fact, another mathematical relationship can be used. The only thing that must be true is that small objects can survive better than large ones. However, large objects have a margin of survival from their own size before they become small. It is therefore a matter of opposing tendencies that maintain the mutual dynamic stability of objects. The principle is also suitable for modelling various economic models.

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CONTACTS:

Jozef Svetlik, Prof. Ing. PhD.
Technical University of Kosice
Faculty of Mechanical Engineering
Department of Production Systems and Robotics
Letna 1/9, 04200 Kosice, Slovak Republic
jozef.svetlik@tuke.sk